

# SYMMETRY AND UNIQUENESS OF NONNEGATIVE SOLUTIONS OF SOME PROBLEMS IN THE HALFSPACE

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December 4, 2012

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## Abstract

We derive some 1-D symmetry and uniqueness or non-existence results for nonnegative solutions of

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N \end{cases}$$

in low dimension, under suitable assumptions on  $A$  and  $g$ . Our method is based upon a combination of Fourier series and Liouville theorems.

**Keywords:** 1-D symmetry for elliptic problems; uniqueness for elliptic problems; Liouville theorems; Fourier series.

## 1 Introduction

This paper concerns symmetry and uniqueness or non-existence of nonnegative solutions to problems of type

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N \end{cases} \quad (1)$$

in low dimension; here  $\operatorname{div}(A(x)\nabla)$  is an elliptic operator (not necessarily uniformly elliptic). As far as  $g$  is concerned, we will see that the existence and the properties of nonnegative solutions to (1) depend strongly on it. With this in mind, we will start considering as model problem

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (2)$$

In section 2, we will prove the following statement.

**Theorem 1.1.** *Let  $N = 2$  or  $3$ . If  $u \in C^2(\overline{\mathbb{R}_+^N})$  solves problem (2) and*

$$\forall M > 0 \quad \exists C(M) > 0 \quad : \quad 0 \leq u(x) \leq C(M) \quad \forall x \in \mathbb{R}^{N-1} \times [0, M], \quad (3)$$

*then*

$$u(x', x_N) = 1 - \cos x_N.$$

This is a result of uniqueness and of 1-D symmetry, i.e. the (unique) solution of (2) is a function depending only on  $x_N$ . Note that assumption (3) means that  $u$  is nonnegative and bounded in every strip of type  $\mathbb{R}^{N-1} \times [0, M]$ .

Unfortunately, we will see that the assumption " $N = 2$  or  $3$ " is substantial for our proof. However, we can still say something for the model problem in higher dimension. This will be the object of subsection 2.1.

As far as the generalization towards problem (1) is concerned, we will see in section 3, Theorem 3.2, that the presence of  $\operatorname{div}(A(x)\nabla)$  instead of the laplacian does not affect the previous result, under suitable assumptions on  $A$ .

A further natural generalization of problem (2) consists in introducing a  $g$  depending only on  $x_N$  instead of the constant function 1:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = u - g(x_N) & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N \end{cases} \quad (4)$$

In this setting, Theorem 4.7 is the counterpart of Theorem 1.1; as an immediate corollary we have

**Corollary 1.2.** *Let  $N = 2$  or  $3$ . Under suitable assumptions on  $A$  and on  $g$ , if  $u \in C^2(\overline{\mathbb{R}_+^N})$  solves (4) and satisfies (3), then  $u$  is uniquely determined and depends only on  $x_N$ .*

Finally, always in section 4, we will show how to use the method developed in the previous sections in order to deal with a wider class of inhomogeneous terms (depending also on  $x'$ ), obtaining sharp results for some particular cases; for instance, we will see that if  $g = g(x')$  and there exists a solution  $u$  of (1) satisfying (3), then  $g$  has to be constant.

The interest in the model problem comes from Berestycki, Caffarelli and Nirenberg: in [1] they proved that a *positive and bounded* solution to (2) does

not exist when  $N \leq 3$ . Their result fits in a wider study of 1-D symmetry and monotonicity for *positive and bounded* solutions to

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases} \quad (5)$$

with  $f$  Lipschitz continuous. If  $N \geq 2$  and  $f(0) \geq 0$ , then a positive and bounded solution is strictly increasing in the  $x_N$  variable (see [1, 3]). Furthermore, always in [1], if  $N \leq 3$ ,  $f \in \mathcal{C}^1(\mathbb{R})$  and  $f(0) \geq 0$ , they showed that a positive and bounded solution depends only on one variable (1-D symmetry). Another contribution, contained in [1], is that the monotonicity and the 1-D symmetry hold true for  $N = 2$  without any restriction on the sign of  $f(0)$ . The proofs of the quoted results are based on the moving planes method and on a previous result in [2], where it is shown that if  $u$  is a positive and bounded solution of (5) and

$$f(M) \leq 0 \quad \text{where} \quad M = \sup_{x \in \mathbb{R}_+^N} u(x),$$

then  $u$  is symmetric and monotone, and  $f(M) = 0$ . We point out that our results are not included in the existing literature, because we are considering *nonnegative* and *not necessarily bounded* solutions, and because in general we are interested in the case  $f(0) < 0$ . In such a situation the moving planes method gives just partial results, as shown by Dancer [4]. We emphasize the fact that the difference between *positive* and *nonnegative* is substantial for  $f(0) < 0$ , since in this case natural solutions are nonnegative and non-monotone, and a positive solution does not necessarily exist; this is clearly the case of the model problem (2). For all these reasons, our approach is different and it is based upon a combination of Fourier series and Liouville theorems.

To complete the essential bibliography for this kind of problems, we mention also the work [6], where symmetry and monotonicity are obtained under weaker regularity assumptions on  $f$ , and an extension in dimension 4 and 5 is given for a wide class of nonlinearities.

**Notation.** We will consider problems in the half space  $\mathbb{R}_+^N := \mathbb{R}^{N-1} \times (0, +\infty)$ . As usual, we will denote by  $(x', x_N)$  a point of  $\mathbb{R}_+^N$ .

The symbols  $\nabla'$ ,  $\text{div}'$  or  $\Delta'$  will be used respectively for the gradient, the divergence or the laplacian in  $\mathbb{R}^{N-1}$ .

The notation  $u_j$  will be used to indicate the partial derivative of  $u$  with respect to the  $x_j$  variable. For any  $x \in \mathbb{R}^N$ , for any  $R > 0$ , we will write  $B_R(x)$  to indicate the ball of centre  $x$  and radius  $R$ . If  $x = 0$ , we will simply write  $B_R$ .

For any  $A \subset \mathbb{R}^N$ ,  $\chi_A$  will denote the characteristic function of  $A$ .

We will use the notation  $\langle \cdot, \cdot \rangle$  for the usual scalar product in any euclidean space. Given a real valued function  $v$ , we denote its positive part as  $v^+$ .

## 2 The model problem

In this section we consider problem (2):

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

We aim at proving Theorem 1.1.

**Remark 1.** Assumption (3) says that  $u$  is nonnegative in the whole  $\mathbb{R}_+^N$  and bounded in every strip of type  $\mathbb{R}^{N-1} \times [0, M]$  (but with arbitrary growth in the  $x_N$ -direction). Assumption (3) is obviously satisfied if  $u$  is nonnegative and bounded. Actually it is sufficient to assume that  $u$  is nonnegative and  $\nabla u$  is bounded, in order to ensure (3). Indeed for every  $M > 0$  we have

$$\begin{aligned} |u(x', x_N)| &= |u(x', x_N) - u(x', 0)| \leq \sup_{\xi \in [0, x_N]} |\nabla u(x', \xi)| x_N \\ &\leq \|\nabla u\|_\infty M = C(M) \quad \forall (x', x_N) \in \mathbb{R}^{N-1} \times [0, M]. \end{aligned}$$

In particular we recover the non-existence result of Berestycki, Caffarelli and Nirenberg.

First we focus on the problem (2) in the strip  $\bar{\Sigma} = \mathbb{R}^{N-1} \times [0, 2\pi]$ ,  $N \geq 2$ . For every  $x' \in \mathbb{R}^{N-1}$ , we denote by  $\tilde{u}(x', \cdot)$  the  $2\pi$ -periodic extension of  $x_N \mapsto u(x', x_N)$ . In view of the smoothness of  $u$ , it follows that the Fourier expansion of  $x_N \mapsto \tilde{u}(x', x_N)$ , i.e.

$$\frac{a_0(x')}{2} + \sum_{m=1}^{+\infty} (a_m(x') \cos(mx_N) + b_m(x') \sin(mx_N)), \quad (6)$$

where

$$\begin{aligned} a_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) \cos(mx_N) dx_N \quad \forall m \geq 0, \\ b_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) \sin(mx_N) dx_N \quad \forall m \geq 1, \end{aligned} \quad (7)$$

is convergent.

Now we determine the equations satisfied by the coefficients above.

**Lemma 2.1.** *Let  $N \geq 2$ . For any  $m \geq 1$  we have*

$$\Delta' a_m(x') = (m^2 - 1)a_m(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)) \quad (8)$$

$$\Delta' b_m(x') = (m^2 - 1)b_m(x') + \frac{m}{\pi} u(x', 2\pi). \quad (9)$$

Also,

$$\Delta' a_0(x') = 2 - a_0(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)).$$

*Proof.* For any  $m \geq 1$  we have

$$\begin{aligned}\Delta' a_m(x') &= \frac{1}{\pi} \int_0^{2\pi} \Delta' u(x', x_N) \cos(mx_N) dx_N \\ &= \frac{1}{\pi} \int_0^{2\pi} (1 - u(x', x_N) - u_{NN}(x', x_N)) \cos(mx_N) dx_N \\ &= -\frac{1}{\pi} \int_0^{2\pi} (u(x', x_N) + u_{NN}(x', x_N)) \cos(mx_N) dx_N.\end{aligned}$$

Integrating by parts twice the last term we obtain

$$\begin{aligned}\Delta' a_m(x') &= -\frac{1}{\pi} \int_0^{2\pi} (u(x', x_N) \cos(mx_N) + m u_N(x', x_N) \sin(mx_N)) dx_N \\ &\quad - \frac{1}{\pi} [u_N(x', x_N) \cos(mx_N)]_{x_N=0}^{2\pi} \\ &= \frac{m^2 - 1}{\pi} \int_0^{2\pi} u(x', x_N) \cos(mx_N) dx_N \\ &\quad - \frac{1}{\pi} (u_N(x', 2\pi) - u_N(x', 0)),\end{aligned}$$

which is equation (8).

With the same procedure we can find equation (9): for any  $m \geq 1$

$$\begin{aligned}\Delta' b_m(x') &= \frac{1}{\pi} \int_0^{2\pi} \Delta' u(x', x_N) \sin(mx_N) dx_N \\ &= \frac{1}{\pi} \int_0^{2\pi} (1 - u(x', x_N) - u_{NN}(x', x_N)) \sin(mx_N) dx_N \\ &= \frac{1}{\pi} \int_0^{2\pi} (-u(x', x_N) \sin(mx_N) + m u_N(x', x_N) \cos(mx_N)) dx_N \\ &= \frac{m^2 - 1}{\pi} \int_0^{2\pi} u(x', x_N) \sin(mx_N) dx_N \\ &\quad + \frac{m}{\pi} [u(x', x_N) \cos(mx_N)]_{x_N=0}^{2\pi}.\end{aligned}$$

As far as  $a_0$  is concerned, we have

$$\begin{aligned}\Delta' a_0(x') &= \frac{1}{\pi} \int_0^{2\pi} \Delta' u(x', x_N) dx_N \\ &= \frac{1}{\pi} \int_0^{2\pi} (1 - u(x', x_N) - u_{NN}(x', x_N)) dx_N \\ &= 2 - \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) dx_N + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)). \quad \square\end{aligned}$$

**Lemma 2.2.** *Both  $b_1$  and  $a_1$  are constant; moreover,*

$$u(x', 2\pi) = 0, \quad u_N(x', 0) = 0 \quad \text{and} \quad u_N(x', 2\pi) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

*Proof.* Using (9) with  $m = 1$  we have

$$\Delta' b_1(x') = \frac{1}{\pi} u(x', 2\pi) \geq 0.$$

Therefore, thanks to (3),  $b_1$  is a subharmonic and bounded function in  $\mathbb{R}^{N-1}$  with  $N = 2$  or  $3$ ; the Liouville theorem for subharmonic functions implies that it is constant, so that in particular  $\Delta' b_1 \equiv 0$ , i.e.  $u(x', 2\pi) = 0$  for every  $x' \in \mathbb{R}^{N-1}$ .

Note that, since  $u \geq 0$  and  $u(x', 2\pi) = 0$ , each  $(x', 2\pi)$  is a point of minimum for  $u$ ; consequently  $u_N(x', 2\pi) = 0$ , and this makes possible to prove that also  $a_1$  is constant: indeed

$$\Delta' a_1(x') = \frac{1}{\pi} u_N(x', 0).$$

Since  $u(x', 0) = 0$  and  $u \geq 0$  in  $\Sigma$ , it follows that  $u_N(x', 0) \geq 0$  for every  $x' \in \mathbb{R}^{N-1}$ . Hence  $a_1$  is a subharmonic and bounded function in  $\mathbb{R}$  or  $\mathbb{R}^2$ , and has to be constant. It follows in particular that  $u_N(x', 0) \equiv 0$  in  $\mathbb{R}^{N-1}$ .  $\square$

An important consequence of the previous Lemma is that the equations for  $a_m$  and  $b_m$  simplify as

$$\Delta' a_m(x') = (m^2 - 1)a_m(x') \quad \forall m \geq 2 \quad (10)$$

$$\Delta' b_m(x') = (m^2 - 1)b_m(x') \quad \forall m \geq 2. \quad (11)$$

Hence, for  $m \geq 2$ ,  $a_m$  and  $b_m$  satisfy an equation of type

$$-\Delta' v(x') + \lambda v(x') = 0 \quad \text{in } \mathbb{R}^{N-1}, \quad (12)$$

with  $\lambda > 0$ . We point out that both  $a_m$  and  $b_m$  are bounded in absolute value in  $\Sigma$  (this follows from assumption (3)).

Bounded solutions of (12) has to vanish identically. This is an immediate consequence of the following general result.

**Lemma 2.3.** *Assume  $N \geq 2$  and let  $v \in C^2(\mathbb{R}^{N-1})$  be a subsolution of*

$$-\Delta' v(x') + c(x')v(x') \leq 0 \quad \text{in } \mathbb{R}^{N-1}, \quad (13)$$

*with  $c(x') \geq \lambda > 0$  in  $\mathbb{R}^{N-1}$ .*

*If  $v^+$  has at most algebraic growth at infinity, then  $v \leq 0$  in  $\mathbb{R}^{N-1}$ ,*

For the proof, it will be useful the following Lemma.

**Lemma 2.4.** *Let  $\vartheta > 0$ ,  $\gamma > 0$ , be such that  $\vartheta < 2^{-\gamma}$ . Let  $R_0 > 0$ ,  $C > 0$  and  $I : (R_0, +\infty) \rightarrow [0, +\infty)$  be such that*

$$\begin{cases} I(R) \leq \vartheta I(2R) & \forall R > R_0 \\ I(R) \leq CR^\gamma & \forall R > R_0. \end{cases} \quad (14)$$

*Then  $I(R) = 0$  for every  $R > R_0$ .*

*Proof.* Iterating the first one of (14) we obtain, for every  $k \in \mathbb{N}$ ,

$$I(R) \leq \vartheta^k I(2^k R) \quad \forall R > R_0.$$

Now the second one gives

$$I(R) \leq C (\vartheta 2^\gamma)^k R^\gamma \quad \forall R > R_0, \forall k \in \mathbb{N}.$$

Since  $0 < \vartheta 2^\gamma < 1$ , letting  $k \rightarrow \infty$  we obtain  $I(R) \leq 0$  for  $R > R_0$ .  $\square$

*Proof of Lemma 2.3.* We introduce a  $\mathcal{C}^\infty$  cut-off function  $\varphi : [0, +\infty) \rightarrow \mathbb{R}$  such that

$$\begin{cases} \varphi(t) = 1 & t \in [0, 1] \\ \varphi(t) = 0 & t \in [2, +\infty) \\ 0 \leq \varphi(t) \leq 1 & t \in (1, 2). \end{cases}$$

We set, for every  $R > 0$ ,  $\varphi_R(x') := \varphi(|x'|/R)$ , which is defined on  $\mathbb{R}^{N-1}$ . Hence

$$\nabla' \varphi_R(x') = \frac{x'}{R|x'|} \varphi' \left( \frac{|x'|}{R} \right).$$

In particular

$$|\nabla' \varphi_R(x')| \leq \frac{C}{R} \chi_{B_{2R}}(x') \quad (15)$$

where  $C$  is a constant independent of  $R$ .

Testing (13) with  $v^+ \varphi_R^2$  we get

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \left( |\nabla' v^+|^2 + \lambda (v^+)^2 \right) \varphi_R^2 &\leq \int_{\mathbb{R}^{N-1}} \left( |\nabla' v^+|^2 + c (v^+)^2 \right) \varphi_R^2 \\ &\leq -2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \langle \nabla' v^+, \nabla' \varphi_R \rangle \leq 2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R |\langle \nabla' v^+, \nabla' \varphi_R \rangle|. \end{aligned} \quad (16)$$

We can use the Cauchy-Schwarz and the Young inequalities: for every  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$\begin{aligned} 2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R |\langle \nabla' v^+, \nabla' \varphi_R \rangle| &\leq 2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R |\nabla' v^+| |\nabla' \varphi_R| \\ &\leq 2\varepsilon \int_{\mathbb{R}^{N-1}} \varphi_R^2 |\nabla' v^+|^2 + 2C_\varepsilon \int_{\mathbb{R}^{N-1}} (v^+)^2 |\nabla' \varphi_R|^2. \end{aligned}$$

Coming back to (16), we obtain

$$\int_{\mathbb{R}^{N-1}} \left( (1 - 2\varepsilon) |\nabla' v^+|^2 + \lambda (v^+)^2 \right) \varphi_R^2 \leq 2C_\varepsilon \int_{\mathbb{R}^{N-1}} (v^+)^2 |\nabla' \varphi_R|^2.$$

Choosing  $\varepsilon < 1/2$  and using the (15), we deduce

$$\int_{B_R} (v^+)^2 \leq \frac{C}{\lambda R^2} \int_{B_{2R}} (v^+)^2.$$

Also, since  $v^+$  has at most algebraic growth at infinity, we have for any  $R > 1$

$$\int_{B_R} (v^+)^2 \leq C' R^{N+2k}$$

for some  $k \geq 0$ ,  $C' > 0$  independent of  $R$ .

We are in position to apply Lemma 2.4, with

$$I(R) := \int_{B_R} (v^+)^2.$$

Here  $\gamma = N + 2k$ ; note that there exists  $R_0 > 1$  such that  $\frac{C'}{\lambda R^2} < 2^{-N-2k}$  for every  $R \geq R_0$ . We set  $\vartheta = \frac{C'}{\lambda R_0^2}$  and we apply Lemma 2.4 to obtain

$$\int_{B_R} (v^+)^2 = 0 \quad \forall R > R_0 \Rightarrow v^+ \equiv 0. \quad \square$$

*Conclusion of the proof of Theorem 1.1.* Applying Lemma 2.3 to equations (10) and (11), we have that the Fourier coefficients  $a_m$  and  $b_m$  are identically 0 for any  $m \geq 2$ . Hence, the Fourier series (6) is reduced to

$$\frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N \quad (17)$$

and, for every  $x \in \bar{\Sigma}$ , it is equal to  $u(x)$ .

The initial condition  $u(x', 0) = 0$  reads

$$\frac{a_0(x')}{2} + a_1 = 0 \Rightarrow a_0 \text{ is constant, equal to } -2a_1.$$

We also proved that  $u_N(x', 0) = 0$ , which implies  $b_1 = 0$ .

Plugging the expression of  $u$  inside the equation  $-\Delta u = u - 1$  we obtain

$$-a_1 \cos x_N + \frac{a_0}{2} + a_1 \cos x_N - 1 = 0 \Rightarrow a_0 = 2,$$

and hence  $a_1 = -1$ .

We proved that if  $u \in \mathcal{C}^2(\bar{\mathbb{R}}_+^N)$  is a solution of (2) satisfying (3) then  $u(x', x_N) = 1 - \cos x_N$  in  $\bar{\Sigma} = \mathbb{R}^{N-1} \times [0, 2\pi]$ .

To extend the result in the whole  $\mathbb{R}_+^N$  we set

$$v_1(x', x_N) := u(x', x_N + 2\pi).$$

It is straightforward to check that  $v_1$  is a nonnegative solution of (2) and satisfies (3), so that it has to coincide with  $1 - \cos x_N$  in  $\bar{\Sigma}$ ; this means that  $u(x', x_N) = 1 - \cos x_N$  for  $(x', x_N) \in \mathbb{R}^{N-1} \times [0, 4\pi]$ . The thesis follows by iteration of this argument.  $\square$



## 2.1 The model problem in higher dimension

In our proof it was crucial the possibility of applying the Liouville Theorem for subharmonic functions, which holds only in  $\mathbb{R}$  and  $\mathbb{R}^2$ . Therefore, despite the fact that our statement seems to be natural in any dimension, we cannot prove it. However, it is still possible to collect some properties of any solution of problem (2) satisfying (3) for  $N \geq 4$ .

We can focus again on the problem in the strip  $\bar{\Sigma}$ , developing  $u$  (or, better, its  $2\pi$ -periodic extension  $\tilde{u}$  in the  $x_N$  variable) in formal Fourier series with respect to  $x_N$ . Note that Lemma 2.1 still holds true. Now, in our analysis the key properties of the solutions was

$$u(x', 2\pi) \equiv 0 \quad \text{and} \quad u_N(x', 0) \equiv 0 \quad \text{in } \mathbb{R}^{N-1}. \quad (18)$$

In this way, equations (8) and (9) are considerably simplified, since all the boundary terms have to vanish identically. This permits to get Theorem 1.1.

**Proposition 2.5.** *Let  $N \geq 2$ . Let  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  be a solution of problem (2) which satisfies (3). Let  $a_m$  and  $b_m$  its formal Fourier coefficients, defined by (7). Assume (18) holds true. Then*

$$u(x', x_N) = 1 - \cos x_N.$$

*Proof.* Under assumption (18), equations (8) and (9) are reduced to

$$\begin{aligned} \Delta' a_m(x') &= (m^2 - 1)a_m(x') & \forall m \geq 1 \\ \Delta' b_m(x') &= (m^2 - 1)b_m(x') & \forall m \geq 1 \end{aligned}$$

(note that  $u_N(x', 2\pi) = 0$  since  $u(x', 2\pi) = 0$  and  $u \geq 0$ ). Hence, Lemma 2.3 implies that  $a_m \equiv 0 \equiv b_m$  for every  $m \geq 2$ , while from the classical Liouville theorem for harmonic function it follows that  $a_1$  and  $b_1$  are constant, so that

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N \quad \text{in } \Sigma.$$

Now we can conclude as in the proof of Theorem 1.1.  $\square$

Also if we cannot prove (18), it is possible to deduce something for the formal Fourier coefficients.

**Proposition 2.6.** *Let  $N \geq 2$ . Let  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  be a solution of problem (2) which satisfies (3). Let  $a_m$  and  $b_m$  its formal Fourier coefficients, defined by (7). Then*

- (i)  $b_m \leq 0$  for every  $m \geq 2$ .
- (ii)  $\frac{b_n}{n} \geq \frac{b_m}{m}$  for every  $n > m \geq 2$ .
- (iii) for every  $m \geq 2$ , either  $b_m < 0$  or  $b_m \equiv 0$  in  $\mathbb{R}^{N-1}$ .

*Proof.* (i) For every  $m \geq 2$  we have

$$-\Delta' b_m(x') + (m^2 - 1)b_m(x') = -\frac{m}{\pi}u(x', 2\pi) \leq 0. \quad (19)$$

From Lemma 2.3, which holds true in any dimension, we deduce that

$$b_m \leq 0 \quad \forall m \geq 2. \quad (20)$$

(ii) For  $m \geq 2$ , let us divide equation (9) by  $m$ :

$$\frac{\Delta' b_m(x')}{m} = \frac{m^2 - 1}{m} b_m(x') + \frac{1}{\pi} u(x', 2\pi).$$

If  $n > m \geq 2$

$$\begin{aligned} -\Delta' \left( \frac{b_m(x')}{m} - \frac{b_n(x')}{n} \right) + (n^2 - 1) \left( \frac{b_m(x')}{m} - \frac{b_n(x')}{n} \right) \\ = (n^2 - m^2) \frac{b_m(x')}{m} \leq 0, \end{aligned}$$

thanks to the fact that  $b_m \leq 0$ . Again, by means of Lemma 2.3, we get

$$\frac{b_n}{n} \geq \frac{b_m}{m} \quad \forall n > m \geq 2.$$

(iii) if there exists  $\bar{x}' \in \mathbb{R}^{N-1}$  such that  $b_m(\bar{x}') = 0$ , the strong maximum principle implies  $b_m \equiv 0$ .  $\square$

It is particularly interesting to observe that, if we know that one  $b_m$  vanishes in one point of  $\mathbb{R}^{N-1}$ , then we can recover Theorem 1.1.

**Corollary 2.7.** *Let  $N \geq 2$ . Let  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  be a solution of problem (2) satisfying (3). Let  $a_m$  and  $b_m$  its formal Fourier coefficients, defined by (7). If there exist  $\bar{m} \geq 2$  and  $\bar{x}' \in \mathbb{R}^{N-1}$  such that  $b_{\bar{m}}(\bar{x}') = 0$ , then  $u(x', x_N) = 1 - \cos x_N$ .*

*Proof.* By point (iii) of the previous Proposition we know that  $b_{\bar{m}} \equiv 0$ . Hence, from (9) for  $\bar{m}$  we get

$$u(x', 2\pi) \equiv 0 \quad \text{in } \mathbb{R}^{N-1}.$$

As a consequence

$$-\Delta' b_m(x') + (m^2 - 1)b_m(x') = 0 \quad \forall m \geq 1,$$

which implies through Lemma 2.3 that  $b_m \equiv 0$  for every  $m \geq 2$ ; also,  $b_1$  turns out to be a bounded harmonic function on the whole  $\mathbb{R}^{N-1}$ , so that it has to be constant. Now we show that  $b_1 = 0$ . Note that

$$\tilde{u}(x', x_N) - b_1 \sin x_N = \frac{a_0(x')}{2} + \sum_{m=0}^{+\infty} a_m(x') \cos(mx_N);$$

hence,  $w(x', x_N) = \tilde{u}(x', x_N) - b_1 \sin x_N$  is an even  $2\pi$ -periodic function in the  $x_N$  variable. Since we are assuming  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  and  $u(x', 2\pi) = 0$ , the function  $w$  is continuous on the whole  $\mathbb{R}$ , and has continuous derivative with respect to  $x_N$ , except at most in  $(x', 0 + 2k\pi)$ , with  $k \in \mathbb{Z}$ . However, the right and left derivatives in these points exist, and in particular

$$w_N(x', 2\pi^-) = u_N(x', 2\pi^-) - b_1 = -b_1.$$

By periodicity and oddness of  $w_N$  it results

$$b_1 = w_N(x', 0^+) = u_N(x', 0^+) - b_1 = u_N(x', 0) - b_1 \Rightarrow u_N(x', 0) = 2b_1.$$

Note that  $u_N(x', 0)$  is constant. Now, plugging this expression in equation (8) with  $m = 1$  we obtain

$$\Delta' a_1(x') = \frac{2b_1}{\pi} \Rightarrow \Delta' \left( a_1(x') - \frac{b_1 x_1^2}{\pi} \right) = 0 :$$

the function  $a_1(x') - b_1 x_1^2/\pi$  is harmonic in the whole  $\mathbb{R}^N$  and has at most algebraic growth with rate 2 (since  $a_1$  is bounded): therefore, the Liouville Theorem implies that

$$a_1(x') = \frac{b_1 x_1^2}{\pi} + P(x'),$$

where  $P$  is a harmonic polynomial. To sum up,  $a_1$  is a bounded polynomial, thus it is constant, which in turns gives  $\Delta' a_1 = 0$ , i.e.  $b_1 = 0$  and finally  $u_N(x', 0) = 0$ . The thesis follows now from Proposition 2.5.  $\square$

### 3 More general operators

In this section we generalize the approach adopted for the model problem to a more general family of elliptic equations (not necessarily uniformly elliptic) obtained by substituting the laplacian with a class of operators in divergence form. To be precise, let  $A(x')$  be a  $N \times N$  matrix of type

$$A(x') = \begin{pmatrix} \widehat{A}(x') & 0 \\ 0 & 1 \end{pmatrix}, \quad (21)$$

where  $\widehat{A}(x')$  is a  $(N-1) \times (N-1)$  symmetric and real matrix with entries  $a_{ij} \in C^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ , such that

$$\forall x' \in \mathbb{R}^{N-1}, \forall \xi \in \mathbb{R}^{N-1} \setminus \{0\} : \sum_{i,j=1}^{N-1} a_{ij}(x') \xi_i \xi_j > 0. \quad (22)$$

Of course, if  $N = 2$  then  $\widehat{A}(x')$  is a scalar positive function.

For the reader's convenience, we recall the following generalization of the classical Liouville theorem, see [5].

**Theorem 3.1.** *Let  $q \geq 0$  and  $B(x) = (b_{ij}(x))$  be a symmetric real matrix, whose entries are  $L^\infty(\mathbb{R}^2)$  functions satisfying:*

$$\text{for a.e. } x' \in \mathbb{R}^2, \forall \xi \in \mathbb{R}^2 \setminus \{0\} : \sum_{i,j=1}^2 b_{ij}(x') \xi_i \xi_j > 0.$$

*Let  $v \in H_{loc}^1(\mathbb{R}^2)$  be a distribution solution of*

$$\begin{cases} -\operatorname{div}(B(x') \nabla v) + q(\langle B(x') \nabla v, \nabla v \rangle) \geq 0 & \text{in } \mathbb{R}^2 \\ v(x') \geq -C & \text{a.e. in } \mathbb{R}^2, \end{cases}$$

*for some positive constant  $C$ . Then  $v$  is a constant function.*

**Remark 2.** It is not difficult to pass from the two dimensional case to the scalar case: it is sufficient to consider a solution of an ODE as a solution of the corresponding PDE.

These results enable us to try to use the arguments of section 2 for the study of

$$\begin{cases} -\operatorname{div}(A(x) \nabla u) = u - 1 & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial \mathbb{R}_+^N. \end{cases} \quad (23)$$

Note that, due to the particular form of  $A$  (cfr. equation (21)), it is reasonable to think that (23) inherits the structure of the model problem solved in the previous section. It is also immediate to check that the function  $1 - \cos x_N$  is, again, a nonnegative solution of (23) satisfying (3). We plan to prove that it is also unique in this class for  $N = 2$  and 3.

**Theorem 3.2.** *Let  $N = 2$  or 3. Let  $A(x')$  be a  $N \times N$  matrix of type (21), where  $\hat{A}(x')$  is a  $(N-1) \times (N-1)$  symmetric and real matrix with entries  $a_{ij} \in C^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ , and such that (22) holds true. If  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  solves (23) and satisfies (3), then*

$$u(x', x_N) = 1 - \cos x_N.$$

We can follow the proof of Theorem 1.1. Again,  $\Sigma = \mathbb{R}^{N-1} \times (0, 2\pi)$ . For every  $x' \in \mathbb{R}^{N-1}$ , we denote by  $\tilde{u}(x', \cdot)$  the  $2\pi$ -periodic extension of  $x_N \mapsto u(x', x_N)$ . As for the model problem, from the smoothness of  $u$  it follows that the Fourier expansion of  $x_N \mapsto \tilde{u}(x', x_N)$  is convergent:

$$\tilde{u}(x', x_N) = \frac{a_0(x')}{2} + \sum_{m=1}^{+\infty} (a_m(x') \cos(mx_N) + b_m(x') \sin(mx_N)),$$

where  $a_m$  and  $b_m$  are defined by (7).

With a slightly modification of the proof of Lemma 2.1, we obtain

**Lemma 3.3.** *Let  $N \geq 2$ . For any  $m \geq 1$  we have*

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) = (m^2 - 1) a_m(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)) \quad (24)$$

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' b_m(x') \right) = (m^2 - 1) b_m(x') + \frac{m}{\pi} u(x', 2\pi). \quad (25)$$

Also,

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_0(x') \right) = 2 - a_0(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)). \quad (26)$$

*Proof.* For any  $m \geq 1$  we have

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) = \frac{1}{\pi} \int_0^{2\pi} \operatorname{div}' \left( \widehat{A}(x') \nabla' u(x', x_N) \right) \cos(mx_N) dx_N. \quad (27)$$

Since  $a_{iN} = a_{Nj} \equiv 0$  for any  $i, j \neq N$ , we have

$$\begin{aligned} \operatorname{div}' \left( \widehat{A}(x') \nabla' u(x', x_N) \right) &= \sum_{i=1}^{N-1} \partial_i \left( \sum_{j=1}^{N-1} a_{ij}(x') u_j(x', x_N) \right) \\ &= \sum_{i=1}^N \partial_i \left( \sum_{j=1}^N a_{ij}(x') u_j(x', x_N) \right) - u_{NN}(x', x_N) \\ &= 1 - u(x', x_N) - u_{NN}(x', x_N). \end{aligned} \quad (28)$$

Hence equation (27) becomes

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) = \frac{1}{\pi} \int_0^{2\pi} (1 - u(x', x_N) - u_{NN}(x', x_N)) \cos(mx_N) dx_N.$$

Now, as usual, we can integrate by parts twice the last term and pass to

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) = (m^2 - 1) a_m(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)),$$

which is (24). The same procedure gives (25) and (26).  $\square$

**Lemma 3.4.** *Both  $b_1$  and  $a_1$  are constant. Moreover,*

$$u(x', 2\pi) = 0, \quad u_N(x', 2\pi) = 0 \quad \text{and} \quad u_N(x', 0) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

*Proof.* In light of the previous Lemma, we have

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' b_1(x') \right) = \frac{1}{\pi} u(x', 2\pi) \geq 0;$$

the function  $b_1$  is bounded (since  $u$  satisfies (3)), and since  $N = 2$  or  $3$  we are in position to apply Theorem 3.1:

$$b_1 = \text{const.} \Rightarrow u(x', 2\pi) = \pi \operatorname{div}' \left( \widehat{A}(x') \nabla' b_1(x') \right) = 0.$$

Note that now  $u_N(x', 2\pi) = 0$ , since  $(x', 2\pi)$  is a point of minimum of  $u$  for every  $x' \in \mathbb{R}^{N-1}$ . Therefore, equation (24) becomes

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_1(x') \right) = \frac{1}{\pi} u_N(x', 0) \geq 0;$$

this means that  $a_1$  satisfies the assumptions of Theorem 3.1:

$$a_1 = \text{const.} \Rightarrow u_N(x', 0) = \pi \operatorname{div}' \left( \widehat{A}(x') \nabla' a_1(x') \right) = 0. \quad \square$$

As a consequence, the equations for  $a_m$  and  $b_m$  simplify:

$$\operatorname{div}' \left( \widehat{A} \nabla' b_m(x') \right) = (m^2 - 1) b_m(x') \quad \forall m \geq 2 \quad (29)$$

$$\operatorname{div}' \left( \widehat{A} \nabla' a_m(x') \right) = (m^2 - 1) a_m(x') \quad \forall m \geq 2. \quad (30)$$

In this way, we proved that for any  $m \geq 2$  both the coefficients  $a_m$  and  $b_m$  are bounded solution of an equation of type

$$-\operatorname{div}' \left( \widehat{A}(x') \nabla' v(x') \right) + \lambda v(x') = 0 \quad \text{in } \mathbb{R}^{N-1}, \quad (31)$$

with  $\lambda > 0$ . In analogy with the model problem, we state the following result.

**Lemma 3.5.** *Assume  $N \geq 2$  and let  $v \in \mathcal{C}^2(\mathbb{R}^{N-1})$  a subsolution of*

$$-\operatorname{div}' \left( \widehat{A}(x') \nabla' v(x') \right) + c(x') v(x') \leq 0 \quad \text{in } \mathbb{R}^{N-1}$$

*with  $c(x') \geq \lambda > 0$ . Here  $\widehat{A}(x')$  is an  $(N-1) \times (N-1)$  matrix with entries  $a_{ij}$  in  $L^\infty(\mathbb{R}^{N-1})$  and such that (22) holds true.*

*If  $v^+$  has at most algebraic growth at infinity, then  $v \leq 0$ .*

*Proof.* For any  $R > 0$ , let  $\varphi_R$  be as in the proof of Lemma 2.3. Recall the (15):

$$|\nabla' \varphi_R(x')| \leq \frac{C}{R} \chi_{B_{2R}}(x').$$

Let us test equation (31) with  $v^+ \varphi_R^2$ :

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \left( \langle \widehat{A}(x') \nabla' v^+, \nabla' v^+ \rangle + c(x') (v^+)^2 \right) \varphi_R^2 \\ = -2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \langle \widehat{A}(x') \nabla' v, \nabla' \varphi_R \rangle. \end{aligned} \quad (32)$$

Under our assumptions on  $\widehat{A}$ , for almost every  $x' \in \mathbb{R}^{N-1}$  the function

$$(\xi_1, \xi_2) \in \mathbb{R}^{2(N-1)} \mapsto \langle \widehat{A}(x') \xi_1, \xi_2 \rangle \in \mathbb{R}$$

defines a bilinear symmetric positive definite form, so that in particular the Cauchy-Schwarz inequality holds true. Hence, using also the Young inequality, we can control the right hand side: for any  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$\begin{aligned} -2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \langle \widehat{A}(x') \nabla' v^+, \nabla' \varphi_R \rangle &\leq 2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \left| \langle \widehat{A}(x') \nabla' v^+, \nabla' \varphi_R \rangle \right| \\ &\leq 2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \sqrt{\langle \widehat{A}(x') \nabla' v^+, \nabla' v^+ \rangle} \sqrt{\langle \widehat{A}(x') \nabla' \varphi_R, \nabla' \varphi_R \rangle} \\ &\leq 2\varepsilon \int_{\mathbb{R}^{N-1}} \varphi_R^2 \langle \widehat{A}(x') \nabla' v^+, \nabla' v^+ \rangle + 2C_\varepsilon \int_{\mathbb{R}^{N-1}} (v^+)^2 \langle \widehat{A}(x') \nabla' \varphi_R, \nabla' \varphi_R \rangle. \end{aligned}$$

Coming back to equation (32), using also the fact that  $c(x') \geq \lambda$ , we find

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \left( (1 - 2\varepsilon) \langle \widehat{A}(x') \nabla' v^+, \nabla' v^+ \rangle + \lambda (v^+)^2 \right) \varphi_R^2 \\ \leq 2C_\varepsilon \int_{\mathbb{R}^{N-1}} (v^+)^2 \langle \widehat{A}(x') \nabla' \varphi_R, \nabla' \varphi_R \rangle. \end{aligned}$$

Choosing  $\varepsilon < 1/2$ , using the assumptions of  $\widehat{A}$  and the estimate (15), we deduce

$$\int_{B_R} (v^+)^2 \leq \frac{C}{\lambda R^2} \int_{B_{2R}} (v^+)^2.$$

Also, since  $v^+$  has at most algebraic growth, we have

$$\int_{B_R} (v^+)^2 \leq C' R^{N+2k}.$$

We can apply Lemma 2.4 again, to find

$$\int_{B_R} (v^+)^2 = 0 \quad \forall R > R_0 \text{ sufficiently large,}$$

which implies  $v^+ \equiv 0$ .  $\square$

*Conclusion of the proof of Theorem 3.2.* The previous Lemma implies that  $a_m \equiv 0$  and  $b_m \equiv 0$  for every  $m \geq 2$ . Therefore, a solution  $u$  of (23) which satisfies (3) has the following expansion in  $\bar{\Sigma}$ :

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N,$$

which is exactly (17). Moreover, we showed that

$$u(x', 0) = 0 \quad \text{and} \quad u_N(x', 0) = 0 \quad \forall x' \in \mathbb{R}^{N-1},$$

hence we can repeat step by step the conclusion of the proof of Theorem 1.1.  $\square$

## 4 More general problems

In this section we apply the previous method to study and classify solutions of

$$\begin{cases} -\operatorname{div}(A(x')\nabla u) = u - g(x', x_N) & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

satisfying (3), when  $N = 2$  or  $3$ . The situation here is much more involved than the one in the previous sections. Indeed, we have to face the occurrence of various phenomena such as: non-existence of solutions and/or the existence and the multiplicity of solutions. Moreover, a solution might not be a function of the  $x_N$  variable only (in fact, if  $g$  depends on  $x'$  such a result cannot be expected). The results that we shall prove, will strongly depend on the form of the function  $g$ .

In what follows, we will always assume that the matrix  $A$  satisfies the assumptions already imposed in the previous section. Therefore, we do not write explicitly these assumptions anymore. Since we are interested in classical solutions, we assume that  $g \in \mathcal{C}(\overline{\mathbb{R}_+^N})$ . We can consider the  $2\pi$ -periodic extension of  $x_N \in (0, 2\pi) \mapsto g(x', x_N)$ : inside  $\Sigma$  we have the expansion

$$\tilde{g}(x', x_N) = \frac{c_0(x')}{2} + \sum_{m=1}^{\infty} (c_m(x') \cos(mx_N) + d_m(x') \sin(mx_N)),$$

where

$$\begin{aligned} c_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} g(x', x_N) \cos(mx_N) dx_N & \forall m \geq 0 \\ d_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} g(x', x_N) \sin(mx_N) dx_N & \forall m \geq 1. \end{aligned} \quad (33)$$

Let us define again  $a_m$  and  $b_m$  by (7); these are the formal Fourier coefficients of  $u$  with respect to the  $x_N$ -variable in  $\Sigma$ . We start writing down the equations satisfied by  $a_m$  and  $b_m$ .

**Lemma 4.1.** *For any  $m \geq 0$  it results*

$$\operatorname{div}'(\hat{A}(x')\nabla' a_m(x')) = (m^2 - 1) a_m(x') + c_m(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)); \quad (34)$$

*For any  $m \geq 1$  it results*

$$\operatorname{div}'(\hat{A}(x')\nabla' b_m(x')) = (m^2 - 1) b_m(x') + d_m(x') + \frac{m}{\pi} u(x', 2\pi). \quad (35)$$

*Proof.* For any  $m \geq 1$ :

$$\begin{aligned} \operatorname{div}'(\hat{A}(x')\nabla' a_m(x')) &= \frac{1}{\pi} \int_0^{2\pi} \operatorname{div}'(\hat{A}(x')\nabla' u(x', x_N)) \cos(mx_N) dx_N \\ &= \frac{1}{\pi} \int_0^{2\pi} (g(x', x_N) - u(x', x_N) - u_{NN}(x', x_N)) \cos(mx_N) dx_N. \end{aligned}$$



Now we can go on with the same computations already developed in Lemma 2.1, with the only difference that

$$\frac{1}{\pi} \int_0^{2\pi} \cos(mx_N) dx_N = 0 \quad \text{while} \quad \frac{1}{\pi} \int_0^{2\pi} g(x', x_N) \cos(mx_N) dx_N = c_m(x').$$

In the end, we obtain

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) = (m^2 - 1) a_m(x') + c_m(x') + \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)).$$

The same procedure gives the equations for  $b_m$  and for  $a_0$ .  $\square$

For a quite general  $g$  the study of these equations does not give a complete classification for the possible solutions of (1). However, in some particular cases we can obtain sharp results. This will be the object of the following subsections.

#### 4.1 Inhomogeneous terms independent of $x_N$

The first generalization concerns a constant  $g$ . It is straightforward to adapt the arguments of the previous sections, obtaining the following result.

**Theorem 4.2.** *Let  $N = 2$  or  $3$ . If  $g(x', x_N) = \theta \in \mathbb{R}$ , one of the following alternatives occurs:*

- (i) *if  $\theta \geq 0$  there exists a unique solution of (1) satisfying (3). This solution is given by*

$$u(x', x_N) = \theta(1 - \cos x_N).$$

- (ii) *if  $\theta < 0$ , problem (1) does not admit any solution satisfying (3).*

The next step in the study is to treat the case  $g = g(x')$ . If we are interested in solutions satisfying (3) and  $g$  is not constant, we can show that we do not have such a kind of solution at all.

**Theorem 4.3.** *Let  $N = 2$  or  $3$ , let  $g = g(x') \in \mathcal{C}(\mathbb{R}^{N-1})$ . If  $g$  is not constant, problem (1) does not admit any solution satisfying (3).*

*Equivalently, if there exists  $u \in \mathcal{C}^2(\overline{R}_+^N)$  which solves (1) and satisfies (3), then  $g$  is constant.*

*Proof.* Assume that  $g$  is not constant; the formal Fourier coefficients of  $g$  are

$$c_0(x') = 2g(x'), \quad c_m(x') \equiv 0 \quad d_m(x') \equiv 0 \quad \forall m \geq 1.$$

By contradiction, let  $u$  be a solution of (1) satisfying (3). Since  $c_1 \equiv 0$  and  $d_1 \equiv 0$ , equations (34) and (35) for  $m = 1$  are

$$\begin{aligned} \operatorname{div}' \left( \widehat{A}(x') \nabla' a_1(x') \right) &= \frac{1}{\pi} (u_N(x', 0) - u_N(x', 2\pi)) \\ \operatorname{div}' \left( \widehat{A}(x') \nabla' b_1(x') \right) &= \frac{1}{\pi} u(x', 2\pi). \end{aligned}$$

Hence we are in position to follow the proof of Lemma 3.4:  $a_1$  and  $b_1$  are constant, and  $u(x', 2\pi), u_N(x', 0), u_N(x', 2\pi) \equiv 0$  in  $\mathbb{R}^{N-1}$ . As a consequence, equations (34) and (35) become

$$\begin{aligned} \operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) &= (m^2 - 1) a_m(x') \\ \operatorname{div}' \left( \widehat{A}(x') \nabla' b_m(x') \right) &= (m^2 - 1) b_m(x'). \end{aligned}$$

Therefore Lemma 3.5 applies:  $a_m = b_m \equiv 0$  for every  $m \geq 2$ , so that

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N.$$

The boundary condition  $u(x', 0) = 0$  implies that  $a_0$  is constant, but (34) for  $m = 0$  yields

$$0 = \operatorname{div}' \left( \widehat{A}(x') \nabla' a_0 \right) = 2g(x') - a_0,$$

a contradiction.  $\square$

## 4.2 A 1-D inhomogeneous term

In this subsection we deal with  $g = g(x_N)$ . In this situation various phenomena may occur. Let us start with :

**Non-existence.** If  $g(x_N) = \sin x_N$ , problem (1) does not admit any solution satisfying (3). This follows from the following general result.

**Proposition 4.4.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\overline{\mathbb{R}_+^N})$  and assume that  $d_1 \geq 0$  in  $\mathbb{R}^{N-1}$ . If there exists a solution  $u$  of (1) such that (3) holds true, then  $d_1 \equiv 0$ ,  $b_1$  is constant,*

$$u(x', 2\pi) = 0 \quad \text{and} \quad u_N(x', 2\pi) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

*Proof.* Let us consider equation (35) for  $m = 1$ : since  $d_1 \geq 0$  and  $u \geq 0$ , we have

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' b_1(x') \right) = d_1 + \frac{1}{\pi} u(x', 2\pi) \geq 0.$$

Due to the boundedness of  $u$  in the strip  $\Sigma$ ,  $b_1$  is bounded in absolute value. Since  $N = 2$  or  $3$ , we can apply Theorem 3.1, obtaining that  $b_1$  is constant, which in turns gives

$$u(x', 2\pi) = -d_1 \Rightarrow u(x', 2\pi) = 0 = d_1 \quad \forall x' \in \mathbb{R}^{N-1},$$

because  $u$  is nonnegative. Note that necessarily  $u_N(x', 2\pi) = 0$ .  $\square$

**Remark 3.** The previous Proposition applies not only if  $g = g(x_N)$ . For instance, it gives analogous non-existence results when

- $g(x', x_N)|_\Sigma$  is decreasing in the  $x_N$  direction ( $g \neq \text{const.}$ ) .

- $g(x', x_N) \geq g(x', 2\pi - x_N)$  for every  $(x', x_N) \in \mathbb{R}^{N-1} \times (0, \pi)$ , with strict inequality in one point.

We have a counterpart of the previous statement which rules out the existence of solutions of (1) satisfying (3) when  $g(x_N) = \cos x_N$ .

**Proposition 4.5.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\overline{\mathbb{R}_+^N})$  and assume that  $d_1 \equiv 0$ ,  $c_1 \geq 0$  in  $\mathbb{R}^{N-1}$ . If there exists a solution  $u$  of (1) such that (3) holds true, then  $c_1 \equiv 0$ ,  $a_1$  is constant, and*

$$u_N(x', 0) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

*Proof.* In light of Proposition 4.4, we know that  $u_N(x', 2\pi) = 0$ . Moreover, as already observed, from  $u(x', 0) = 0$  and  $u \geq 0$  it follows  $u_N(x', 0) \geq 0$ . Thus, considering equation (34) for  $m = 1$ , we get

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_1(x') \right) = c_1 + \frac{1}{\pi} u_N(x', 0) \geq 0,$$

since  $c_1 \geq 0$ . The function  $a_1$  is bounded in absolute value, hence for Theorem 3.1 it is constant. Therefore

$$c_1 + u_N(x', 0) = 0 \Rightarrow u_N(x', 0) = 0 = c_1 \quad \forall x' \in \mathbb{R}^{N-1}. \quad \square$$

**Existence and multiplicity.** For every  $N \geq 2$ ,

$$u_A(x', x_N) = x_N + A \sin x_N, \quad A \in [-1, 1],$$

is a one-parameter family of solutions of (1) with  $g(x_N) = x_N$ ; each  $u_A$  satisfies (3). Note that in this case  $c_1 = 0$  while  $d_1 < 0$ , so that the previous Propositions do not apply. Note also that  $u_A$  is unbounded in  $\mathbb{R}_+^N$  for every  $A \in [-1, 1]$ .

**Existence, uniqueness and 1-D symmetry.** For  $m \geq 2$ , the function  $u(x', x_N) = \frac{1}{m^2-1} (1 - \cos(mx_N))$  is the *unique* solution, satisfying (3), of problem (1) for  $g(x_N) = \frac{1}{m^2-1} + \cos(mx_N)$ . Furthermore,  $u$  has 1-D symmetry. The uniqueness result is a consequence of the following general result.

**Theorem 4.6.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\mathbb{R})$  be such that*

$$c_1 \geq 0 \quad \text{and} \quad d_1 \geq 0, \quad (36)$$

*where  $c_m$  and  $d_m$  are the Fourier coefficients of the function  $g$  in  $(0, 2\pi)$ , defined by (33). If there exists  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  which solves problem (1) and satisfies (3), then necessarily  $c_1 = d_1 = 0$ . In this case, the restriction of  $u$  to  $\overline{\Sigma}$  is 1-dimensional and is uniquely determined as the solution of*

$$\begin{cases} -u''(x_N) = u(x_N) - g(x_N) & \text{in } (0, 2\pi) \\ u(0) = u(2\pi) = 0 \\ u'(0) = u'(2\pi) = 0. \end{cases} \quad (37)$$

In particular, in  $\bar{\Sigma}$  we have

$$u(x', x_N) = \frac{c_0}{2} + \left( -\frac{c_0}{2} + \sum_{m=2}^{+\infty} \frac{c_m}{m^2 - 1} \right) \cos x_N + \left( \sum_{m=2}^{+\infty} \frac{m}{m^2 - 1} d_m \right) \sin x_N \\ - \sum_{m=2}^{+\infty} \left( \frac{c_m}{m^2 - 1} \cos(mx_N) + \frac{d_m}{m^2 - 1} \sin(mx_N) \right). \quad (38)$$

*Proof.* From Propositions 4.4 and 4.5 we know that, if  $u$  exists, then  $c_1$  and  $d_1$  has to be 0; in this case  $a_1$  and  $b_1$  are constant, and  $u(x', 2\pi), u_N(x', 0), u_N(x', 2\pi) = 0$  in  $\mathbb{R}^{N-1}$ . Therefore, equations (34) and (35) for  $m \geq 2$  simplify as

$$\begin{aligned} \operatorname{div}' \left( \hat{A}(x') \nabla' a_m(x') \right) &= (m^2 - 1) a_m(x') + c_m \\ \operatorname{div}' \left( \hat{A}(x') \nabla' b_m(x') \right) &= (m^2 - 1) b_m(x') + d_m, \end{aligned}$$

i.e.

$$\begin{aligned} -\operatorname{div}' \left( \hat{A}(x') \nabla' \left( a_m(x') + \frac{c_m}{m^2 - 1} \right) \right) + (m^2 - 1) \left( a_m(x') + \frac{c_m}{m^2 - 1} \right) &= 0 \\ -\operatorname{div}' \left( \hat{A}(x') \nabla' \left( b_m(x') + \frac{d_m}{m^2 - 1} \right) \right) + (m^2 - 1) \left( b_m(x') + \frac{d_m}{m^2 - 1} \right) &= 0. \end{aligned}$$

We can apply Lemma 3.5, obtaining

$$\begin{aligned} a_m(x') &= a_m = -\frac{c_m}{m^2 - 1} & \forall m \geq 2 \\ b_m(x') &= b_m = -\frac{d_m}{m^2 - 1} & \forall m \geq 2. \end{aligned}$$

Now, let us consider in  $\Sigma$

$$\begin{aligned} \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N \\ - \sum_{m=2}^{+\infty} \left( \frac{c_m}{m^2 - 1} \cos(mx_N) + \frac{d_m}{m^2 - 1} \sin(mx_N) \right). \end{aligned}$$

It is a series of  $\mathcal{C}^\infty$  functions which is convergent together with the series of the derivates w.r.t.  $x_N$ , since the sequences  $\{c_m\}$  and  $\{d_m\}$  belong to  $l^2$ . In  $\Sigma$  the series is equal to  $u$ , and the equality can be extended up to the boundary since both the series itself and  $u$  are  $\mathcal{C}^1(\bar{\Sigma})$ . We also know that  $u(x', 0) = 0$  and  $u_N(x', 0) = 0$ . Using the Dirichlet boundary condition we get that  $a_0$  is constant too, and in particular equation (34) for  $m = 0$  implies  $a_0 = c_0$ . Now, from the "initial" conditions we get the expression of  $a_1$  and  $b_1$ .

To sum up, we proved that  $u|_\Sigma$  is 1-D, thus a solution of

$$-u''(x_N) = u(x_N) - g(x_N) \quad \text{for } x_N \in (0, 2\pi)$$

with the boundary conditions stated in (37).  $\square$

As an immediate consequence we obtain

**Theorem 4.7.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\mathbb{R})$  be a  $2\pi$ -periodic function satisfying (36), where  $c_m$  and  $d_m$  are the Fourier coefficients of the function  $g$ . If there exists  $u \in \mathcal{C}^2(\overline{\mathbb{R}_+^N})$  which solves problem (1) and satisfies (3), then necessarily  $c_1 = d_1 = 0$ . In this case,  $u$  is 1-dimensional,  $2\pi$ -periodic and it is uniquely determined as the solution of*

$$\begin{cases} -u''(x_N) = u(x_N) - g(x_N) & \text{in } (0, +\infty) \\ u(0) = u(2\pi) = 0 \\ u'(0) = u'(2\pi) = 0. \end{cases}$$

The expression of  $u$  in Fourier series is given by (38).

In view of the example with  $g(x_N) = x_N$  ( $c_1 = 0$ ,  $d_1 < 0$ ) we see that the assumptions  $c_1 \geq 0$  and  $d_1 \geq 0$  are necessary for Theorem 4.6 and Theorem 4.7. We also remark that the non-negativity of both  $c_1$  and  $d_1$  is not sufficient to guarantee the existence of a solution of (1) which satisfies (3). Indeed, as an immediate consequence of Proposition 4.11 (proved in the next subsection), we have non-existence of solutions of (1) satisfying (3) in case

$$g(x_N) = C_1 \sin(mx_N) \quad \text{or} \quad g(x_N) = C_2 \cos(mx_N) \quad m \geq 2, C_1, C_2 \in \mathbb{R}.$$

Note that  $c_1 = d_1 = 0$  in the above examples.

Another class of functions  $g$  for which there is non existence is considered in the next result, of independent interest,

**Proposition 4.8.** *Let  $N \geq 2$ . If  $g \leq 0$  and it is non-constant, then a nonnegative solution of (1) has to be positive. In particular, if  $N = 2, 3$  and  $d_1 \geq 0$ , then problem (1) does not admit any solution satisfying (3).*

*Proof.* By the strong maximum principle  $u$  must be positive in  $\mathbb{R}_+^N$ . Since  $d_1 \geq 0$ , if a solution  $u$  existed, from Proposition 4.4 it should satisfy  $u(x', 2\pi) = 0$  for every  $x' \in \mathbb{R}^{N-1}$ . A contradiction.  $\square$

A typical example is given by the function  $g(x_N) = -\theta - \cos x_N$ , with  $\theta \geq 1$ . Note that  $c_1 < 0$  and  $d_1 = 0$  in this example.

### 4.3 General inhomogeneous terms

In this subsection we will consider  $g$ -s depending on both  $x'$  and  $x_N$ . As before, we will denote by  $c_m$  and  $d_m$  the Fourier coefficient of the  $2\pi$ -periodic extension of  $x_N \in (0, 2\pi) \mapsto g(x', x_N)$ .

We have always begun our analysis trying to prove that

$$u(x', 2\pi) \equiv 0 \quad \text{and} \quad u_N(x', 0) \equiv 0 \quad \text{in } \mathbb{R}^{N-1}. \quad (39)$$

As a consequence, equations (34) and (35) are considerably simplified, since all the boundary terms have to vanish identically:

$$\begin{aligned} \operatorname{div}' \left( \widehat{A}(x') \nabla' a_m(x') \right) &= (m^2 - 1) a_m(x') + c_m(x') \\ \operatorname{div}' \left( \widehat{A}(x') \nabla' b_m(x') \right) &= (m^2 - 1) b_m(x') + d_m(x'). \end{aligned}$$

We have already observed that, if  $N = 2$  or  $3$ , sufficient conditions in order to obtain (39) are  $c_1 \geq 0$  and  $d_1 \geq 0$ .

In general (for every  $N \geq 2$ ), assume that (39) holds true. Assume also that there exists  $\bar{m} \geq 2$  such that  $c_{\bar{m}} \equiv 0$ . Then

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_{\bar{m}}(x') \right) = (\bar{m}^2 - 1) a_{\bar{m}}(x'),$$

which is of type (31) with  $\lambda > 0$ . From Lemma 3.5 it follows  $a_{\bar{m}} \equiv 0$ . The same holds true for every  $b_{\bar{m}}$  such that  $d_{\bar{m}} \equiv 0$ . We point out that this is true even for  $N > 3$ .

**Proposition 4.9.** *Let  $N \geq 2$ , let  $g \in \mathcal{C}(\overline{\mathbb{R}}_+^N)$ , let  $u$  be a solution of (1) satisfying (3), and let  $a_m$  and  $b_m$  be its formal Fourier coefficients in  $\Sigma$  defined by (7); assume that (39) holds true. Then for every  $m \geq 2$  such that  $c_m \equiv 0$  it results  $a_m \equiv 0$ , and for every  $m \geq 2$  such that  $d_m \equiv 0$  it results  $b_m \equiv 0$ .*

As far as the coefficient  $a_0$  is concerned, we have a similar result, but only in low dimension.

**Proposition 4.10.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\overline{\mathbb{R}}_+^N)$ , let  $u$  be a solution of (1) satisfying (3), and let  $a_m$  and  $b_m$  be its formal Fourier coefficients defined by (7); assume that (39) holds true. If  $c_0 \leq 0$ , then  $a_0 = c_0 \equiv 0$ .*

*Proof.* Since  $u \geq 0$ , equation (34) for  $m = 0$  is

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_0(x') \right) = -a_0(x') + c_0(x') \leq 0 \quad \text{in } \mathbb{R}^{N-1}.$$

Since  $a_0$  is bounded and  $N = 2$  or  $3$ , for Theorem 3.1  $a_0$  is constant. But then

$$0 = \operatorname{div}' \left( \widehat{A}(x') \nabla' a_0(x') \right) = -a_0 + c_0.$$

Thus,  $0 \leq a_0 = c_0 \leq 0$ . □

In what follows we first consider

$$g(x', x_N) = f(x') \varphi(x_N) \in \mathcal{C}(\overline{\mathbb{R}}_+^N), \quad g \not\equiv 0.$$

In the expansion of the  $2\pi$ -periodic extension of  $x_N \in (0, 2\pi) \mapsto g(x', x_N)$ , the Fourier coefficients are

$$c_m(x') = f(x') \gamma_m \quad \forall m \geq 0, \quad d_m(x') = f(x') \delta_m \quad \forall m \geq 1,$$

where  $\gamma_m$  and  $\delta_m$  are the (constant) Fourier coefficients of the  $2\pi$ -periodic extension of  $x_N \in (0, 2\pi) \mapsto \varphi(x_N)$ .

**Remark 4.** Let  $N = 2$  or  $3$ . In light of Propositions 4.4 and 4.5, we know that if  $f(x')\delta_1 \geq 0$ ,  $f(x')\delta_1 \neq 0$ , then there are no solutions of (1) satisfying (3). The same holds true if  $f(x')\gamma_1 \geq 0$ ,  $f(x')\gamma_1 \neq 0$  and  $\delta_1 = 0$ .

If  $N = 2$  or  $3$  and  $\gamma_1 = \delta_1 = 0$ , from Propositions 4.4 and 4.5 we know that  $a_1$  and  $b_1$  are constant and (39) holds true. Hence, equations (34) and (35) simplify as

$$\begin{aligned} \operatorname{div}' \left( \widehat{A}(x') \nabla' a_m \right) &= (m^2 - 1)a_m + f(x')\gamma_m \quad \forall m \neq 1, \\ \operatorname{div}' \left( \widehat{A}(x') \nabla' b_m \right) &= (m^2 - 1)b_m + f(x')\delta_m \quad \forall m \geq 2. \end{aligned}$$

It is not difficult to obtain the following non-existence result.

**Proposition 4.11.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\overline{\mathbb{R}}_+^N)$  and assume*

$$g(x', x_N) = f(x') \cos(mx_N) \quad \text{or} \quad g(x', x_N) = f(x') \sin(mx_N),$$

where  $m \geq 2$  and  $f$  is not identically 0. Then there are no solutions of (1) satisfying (3).

*Proof.* Let us first consider the case  $g(x', x_N) = f(x') \sin(mx_N)$ . By contradiction, let  $u$  be a solution of (1) satisfying (3). Applying Propositions 4.4, 4.5, 4.9 and 4.10, we obtain the following particular form for  $u$  in  $\Sigma$ :

$$u(x', x_N) = a_1 \cos x_N + b_1 \sin x_N + b_m(x') \sin(mx_N).$$

Since  $u_N(x', 0) \equiv 0$ , we deduce that  $b_m = -\frac{b_1}{m}$  is constant. On the other hand it is a solution of

$$0 = \operatorname{div}' \left( \widehat{A}(x') \nabla' b_m \right) = (m^2 - 1)b_m + f(x');$$

thus  $f$  must be constant, that is  $f(x') = f \equiv \theta \in \mathbb{R} \setminus \{0\}$ . But in this case, imposing the initial condition  $u(x', 0) = 0$  we would obtain  $a_1 = 0$  and consequently

$$u(x', x_N) = \frac{\theta}{m^2 - 1} (m \sin x_N - \sin(mx_N)),$$

which does not satisfy (3) because it assumes negative values (it is odd,  $2\pi$ -periodic and not identically zero).

When  $g(x', x_N) = f(x') \cos(mx_N)$ , we can argue as before to find that  $u$  has the form :

$$u(x', x_N) = a_1 \cos x_N + b_1 \sin x_N + a_m(x') \cos(mx_N).$$

The boundary condition  $u_N(x', 0) = 0$  implies  $b_1 = 0$ , while we get  $a_m(x') = -a_1 = \text{const.}$  from  $u(x', 0) = 0$ . Hence  $0 = (m^2 - 1)a_m + f(x')$  and so  $f(x') = f \equiv \theta \in \mathbb{R} \setminus \{0\}$ . Finally  $u$  has the form

$$u(x', x_N) = \frac{\theta}{m^2 - 1} (\cos(x_N) - \cos(mx_N)).$$

Observe that  $0 \leq u(x', \frac{2\pi}{m}) = \frac{\theta}{m^2 - 1} (\cos(\frac{2\pi}{m}) - 1)$  implies  $\theta \leq 0$ , while  $0 \leq u(x', \frac{\pi}{m}) = \frac{\theta}{m^2 - 1} (\cos(\frac{\pi}{m}) + 1)$  yields  $\theta \geq 0$ . A contradiction.  $\square$

More in general, the same proof yields

**Proposition 4.12.** *Let  $N = 2$  or  $3$ , let  $g \in \mathcal{C}(\overline{\mathbb{R}}_+^N)$  and assume*

$$g(x', x_N) = \frac{c_0(x')}{2} + \sum_{m \in I_1} c_m(x') \cos(mx_N) + d_{\bar{n}}(x') \sin(\bar{n}x_N)$$

$$\text{or } g(x', x_N) = c_{\bar{m}}(x') \cos(\bar{m}x_N) + \sum_{n \in I_2} d_n(x') \sin(nx_N),$$

where  $I_1, I_2 \subset (\mathbb{N} \setminus \{0, 1\})$  are finite sets,  $\bar{n} \geq 2$ ,  $\bar{m} \in (\mathbb{N} \setminus \{1\})$  and  $d_{\bar{n}}, c_{\bar{m}}$  are not identically constant. Then there are no solutions of (1) satisfying (3).

In what follows we set  $N = 2$  or  $3$  and we show that it is possible to use the method of the Fourier coefficients in order to obtain a complete classification when  $c_1 = d_1 = 0$  and only a finite number of the Fourier coefficients of  $g$  are not identically zero.

Let

$$g(x', x_N) = \frac{c_0(x')}{2} + \sum_{m \in I_1} c_m(x') \cos(mx_N) + \sum_{n \in I_2} d_n(x') \sin(nx_N), \quad (40)$$

where  $I_1 = \{m_1, \dots, m_{k_1}\}, I_2 = \{n_1, \dots, n_{k_2}\} \subset (\mathbb{N} \setminus \{0, 1\})$ . As far as  $c_0$  is concerned, it can be identically 0 or not. Only to fix our minds, we assume  $c_0(x') \neq 0$ ; furthermore, for the sake of simplicity, we suppose that  $c_0, c_{m_j}, d_{n_j} \in \mathcal{C}^\infty(\mathbb{R}^{N-1})$ .

In what follows we will show that, if there exists  $u \in \mathcal{C}^2(\overline{\mathbb{R}}_+^N)$  which solves (1) for this particular  $g$  and satisfies (3), then we can determine the explicit expression of  $u$ .

Note that, since  $c_1 = d_1 = 0$ , Propositions 4.4 and 4.5 imply that  $a_1$  and  $b_1$  are constant, and (39) holds true; thus, by Proposition 4.9 we obtain

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N$$

$$+ \sum_{j=1}^{k_1} a_{m_j}(x') \cos(m_j x_N) + \sum_{j=1}^{k_2} b_{n_j}(x') \sin(n_j x_N),$$

in  $\Sigma$ , where  $a_0, a_{m_j}$  and  $b_{n_j}$  are solutions of

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_0(x') \right) = -a_0(x') + c_0(x') \quad (41)$$

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' a_{m_j}(x') \right) = (m_j^2 - 1)a_{m_j}(x') + c_{m_j}(x') \quad (42)$$

$$\operatorname{div}' \left( \widehat{A}(x') \nabla' b_{n_j}(x') \right) = (n_j^2 - 1)b_{n_j}(x') + d_{n_j}(x'). \quad (43)$$



Propositions 4.11 and 4.12 imply that, if there exists a unique  $m \in \mathbb{N} \setminus \{1\}$  such that  $c_m \neq 0$  and is not constant, or if there exists a unique  $m \geq 2$  such that  $d_m \neq 0$  and is not constant, then a solution of (1) satisfying (3) does not exist. If we are not in this situation and such a solution exists, this system of PDEs (or ODEs if  $N = 2$ ), together with the boundary conditions  $u(x', 0) = 0$  and  $u_N(x', 0) = 0$  permits to deduce the explicit expression of  $a_0, a_{m_j}$  and  $b_{n_j}$ . We start observing that the boundary condition  $u(x', 0) = 0$  involves only  $a_0$  and  $a_{m_j}$ , while  $u_N(x', 0) = 0$  involves the  $b_{n_j}$ . Thus, we can consider the system of  $k_1 + 2$  equations given by  $u(x', 0) = 0$  together with (41) and (42); the unknowns are the functions  $a_0$  and  $a_{m_j}$ , while we consider  $a_1$  as a parameter; from  $u(x', 0) = 0$  we get

$$a_0(x') = -2a_1 - 2 \sum_{j=1}^{k_1} a_{m_j}(x'); \quad (44)$$

As a consequence

$$\operatorname{div}' \left( \hat{A}(x') \nabla' a_0(x') \right) = -2 \sum_{j=1}^{k_1} \operatorname{div}' \left( \hat{A}(x') \nabla' a_{m_j}(x') \right),$$

and

$$-a_0(x') + c_0(x') = 2a_1 + 2 \sum_{j=1}^{k_1} a_{m_j}(x') + c_0(x'),$$

so that equation (41) gives

$$\sum_{j=1}^{k_1} \operatorname{div}' \left( \hat{A}(x') \nabla' a_{m_j}(x') \right) = -a_1 - \sum_{j=1}^{k_1} a_{m_j}(x') - \frac{c_0(x')}{2}.$$

We plug (42) for  $j \geq 1$  on the left hand side:

$$\sum_{j=1}^{k_1} [(m_j^2 - 1)a_{m_j}(x') + c_{m_j}(x')] = -a_1 - \sum_{j=1}^{k_1} a_{m_j}(x') - \frac{c_0(x')}{2},$$

i.e.

$$a_{m_1}(x') = \frac{1}{m_1^2} \left[ -a_1 - f(x') - \sum_{j=2}^{k_1} m_j^2 a_{m_j}(x') \right], \quad (45)$$

where  $f(x') = c_0(x')/2 + \sum_{j=1}^{k_1} c_{m_j}(x')$ . Note that now equation (45) together with (42) for  $j \geq 2$  is a system of  $k_1 + 1$  equations in the unknowns  $a_{m_j}$  but without  $a_0$ . If we can solve it, we can recover  $a_0$  using the (44).

We can iterate the same argument: from (45) we have

$$\begin{aligned} & \operatorname{div}' \left( \hat{A}(x') \nabla' a_{m_1}(x') \right) \\ &= \frac{1}{m_1^2} \left[ -\operatorname{div}' \left( \hat{A}(x') \nabla' f(x') \right) - \sum_{j=2}^{k_1} m_j^2 \operatorname{div}' \left( \hat{A}(x') \nabla' a_{m_j}(x') \right) \right] \end{aligned}$$

(this is why we required the  $c_{m_j}$  smooth) and

$$\begin{aligned} (m_1^2 - 1)a_{m_1}(x') + c_{m_1}(x') \\ = \frac{m_1^2 - 1}{m_1^2} \left[ -a_1 - f(x') - \sum_{j=2}^{k_1} m_j^2 a_{m_j}(x') \right] + c_{m_1}(x'); \end{aligned}$$

equation (42) for  $j = 1$  gives

$$\begin{aligned} -\operatorname{div}' \left( \hat{A}(x') \nabla' f(x') \right) - \sum_{j=2}^{k_1} m_j^2 \operatorname{div}' \left( \hat{A}(x') \nabla' a_{m_j}(x') \right) \\ = -(m_1^2 - 1)a_1 - (m_1^2 - 1)f(x') - (m_1^2 - 1) \sum_{j=2}^{k_1} m_j^2 a_{m_j}(x') + m_1^2 c_{m_1}(x'), \end{aligned}$$

i.e.

$$\begin{aligned} a_{m_2}(x') = \frac{1}{m_2^2(m_2^2 - m_1^2)} \\ \cdot \left[ (m_1^2 - 1)a_1 - f_1(x') - \sum_{j=3}^{k_1} m_j^2(m_j^2 - m_1^2)a_{m_j}(x') \right], \quad (46) \end{aligned}$$

where

$$f_1(x') = -\operatorname{div}' \left( \hat{A}(x') \nabla' f(x') \right) - \sum_{j=2}^{k_1} m_j^2 c_{m_j}(x') - c_{m_1}.$$

Equation (46) together with (42) for  $j \geq 2$  is a system of  $k_1$  equations in the unknowns  $a_{m_j}$  for  $j \geq 2$ , but without  $a_0$  and  $a_{m_1}$ . If we can solve it, we can recover  $a_{m_1}$  using the (45), and then  $a_0$  using the (44).

Iterating the procedure  $k_1 + 2$  times (here we have to assume  $k_1$  finite), we obtain  $a_{m_{k_1}}$  as function of the Fourier coefficients of the  $g$  (note that the more  $k_1$  is large the more we have to require the  $c_{m_j}$ -s smooth), and successively the others  $a_{m_j}$ . Note that  $a_0$  and  $a_{m_j}$  are functions of  $a_1$ .

The same procedure works for the coefficients  $b_{n_j}$ -s, starting from  $u_N(x', 0) = 0$ . In the end we get the explicit expression of  $u$  in function of the two “parameters”  $a_1$  and  $b_1$ . At this point it is sufficient to impose that  $u$  solves the considered differential equation to determine  $a_1$  and  $b_1$ .

Let us see the iterative procedure in action with an example: let  $N = 2$  and

$$\begin{aligned} g(x, y) &= \left( \frac{2}{(1+x^2)^2} - 4 \frac{x}{(1+x^2)^2} \arctan x + (\arctan x)^2 \right) \\ &\quad + \left( -\frac{2}{(1+x^2)^2} + 4 \frac{x}{(1+x^2)^2} \arctan x + 3(\arctan x)^2 \right) \cos(2y) \\ &= \frac{c_0(x)}{2} + c_2(x) \cos(2y). \quad (47) \end{aligned}$$

**Proposition 4.13.** *There is a unique solution of*

$$\begin{cases} -\Delta u = u - g & \text{in } \mathbb{R}_+^N \\ u(x, 0) = 0 \\ u \text{ satisfies (3),} \end{cases} \quad (48)$$

whose explicit expression is

$$u(x, y) = (\arctan x)^2 (1 - \cos(2y)).$$

*Proof.* Due to the form of  $g$  we know that if  $u$  solves (48) then

$$u(x, y) = \frac{a_0(x)}{2} + a_1 \cos y + b_1 \sin y + a_2(x) \cos(2y), \quad (49)$$

with  $u_y(x, 0) = 0$  (Lemma 3.4 and Proposition 4.9). Thus  $b_1 = 0$ . As far as  $a_0$  and  $a_2$  is concerned, they solve

$$a_0''(x) = -a_0(x) + c_0(x) \quad (50)$$

$$a_2''(x) = 3a_2(x) + c_2(x). \quad (51)$$

From  $u(x, 0) = 0$  we deduce

$$a_0(x) = -2a_1 - 2a_2(x). \quad (52)$$

Hence (50) gives

$$a_2''(x) = -a_1 - a_2(x) - \frac{c_0(x)}{2};$$

we plug (51) on the left hand side, obtaining

$$a_2(x) = -\frac{c_0(x)}{8} - \frac{c_2(x)}{4} - \frac{a_1}{4} = -(\arctan x)^2 - \frac{a_1}{4}, \quad (53)$$

and consequently from (52)

$$a_0(x) = \frac{c_0(x)}{4} + \frac{c_2(x)}{2} - \frac{3}{2}a_1 = 2(\arctan x)^2 - \frac{3}{2}a_1. \quad (54)$$

Note that it is sufficient to substitute the explicit expressions of  $c_0$  and  $c_2$  (which are given by  $g$ ) in order to get  $a_0$  and  $a_2$ , and no integration is required. So far, we proved that a solution of (48) is of type

$$\begin{aligned} u_{a_1}(x, y) &= (\arctan x)^2 (1 - \cos(2y)) - \frac{a_1}{4} (3 - 4 \cos y + \cos(2y)) \\ &= (\arctan x)^2 (1 - \cos(2y)) - \frac{a_1}{2} (1 - \cos y)^2, \end{aligned}$$

which is non negative if and only if  $a_1 \leq 0$ . It is straightforward to check that  $u_{a_1}$  solves (48) only if  $a_1 = 0$ .  $\square$

**Remark 5.** For a generic  $g$  of the form (40), the iterative procedure we introduced above can be used as a test in order to check if (48) has at least one solution satisfying (3).

For instance it is immediate to check that (48) with

$$g(x, y) = \cos(2x) + \sin(3x) \cos(2y),$$

has not a solution satisfying (3). Indeed, if such a solution existed, then its explicit expression would be (49) with  $a_0$  and  $a_2$  given by

$$a_0(x) = \frac{c_0(x)}{4} + \frac{c_2(x)}{2} - \frac{3}{2}a_1 \quad a_2(x) = -\frac{c_0(x)}{8} - \frac{c_2(x)}{4} - \frac{a_1}{4},$$

(cf. (54) and (53)) where  $c_0(x) = 2 \cos(2x)$  and  $c_2(x) = \sin(3x)$ ; but  $\frac{a_0(x)}{2} + a_1 \cos(y) + a_2(x) \cos(2y)$  is not a solution of  $-\Delta u = u - g$ .

Last but not least, we also remark that if  $\lambda_1, \dots, \lambda_k$  are nonnegative real numbers and  $u_1, \dots, u_k$  are solutions of (48) with  $g = g_j$ ,  $j = 1, \dots, k$ , then the function  $u = \sum_{j=1}^k \lambda_j u_j$  is a solution of (48) with  $g = \sum_{j=1}^k \lambda_j g_j$ . Thus, combining in a suitable way the examples considered before, we can construct many other functions  $g$  for which we have existence and uniqueness of the solution or existence and multiplicity of the solutions.

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